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Explicit solutions for the coupled stretching–bending problems of holes in composite laminates

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Abstract

Although the classical lamination theory was developed long time ago, it is still not easy to apply this theory to find the analytical solutions for the curvilinear boundary value problems especially when the stretching and bending are coupled each other. To overcome the difficulties, recently we developed a Stroh-like formalism for the general composite laminates. By using this formalism, most of the relations for the coupled stretching–bending problems can be organized into the forms of Stroh formalism for two-dimensional anisotropic elasticity problems. With this newly developed Stroh-like formalism, it becomes easier to obtain an analytical solution for the coupled stretching–bending problems of holes in composite laminates. Because the Stroh-like formalism is a complex variable formalism, the analytical solutions for the whole field are expressed in complex form. Through the use of some identities derived in this paper, the resultant forces and moments around the hole boundary are obtained explicitly in real form. Due to the lack of analytical solutions for the general cases, the comparison is made with the existing analytical solutions for some special cases. In addition, to show the generality of our analytical solutions, several numerical examples are presented to discuss the coupling effect of the laminates and the shape effect of the holes.

Keywords: Composite laminates; Lamination theory; Hole; Stroh formalism; Coupled stretching–bending analysis

1. Introduction

The problems of composite laminates containing holes have been studied extensively for two-dimensional problems. Although the classical lamination theory was developed long time ago (Jones, 1974), it is not easy to apply this theory to find the analytical solutions for the problems with curvilinear boundaries, like the hole problems. Even the problems of composite laminates with holes are very important in engineering applications and have been solved vastly in two-dimensional problems, it is rarely solved when the laminates make the in-plane and plate bending problems couple each other subjected to in-plane forces and/or out-of-plane bending moments. Searching for the literature, the only

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related analytical solutions for the plate bending problems are obtained by Lekhnitskii (1938) who solved the orthotropic plates weakened by a circular hole by using Lekhnitskii's complex variable method (Lekhnitskii, 1968). Although Lu and Mahrenholtz (1994) solved the general anisotropic plates containing a polygon-like hole by using the modified Stroh's complex variable formalism, the eigen-relation derived in their paper is not in the form of Stroh formalism and no verification is provided in their paper. Hsieh and Hwu (2002a) solved the anisotropic plate weakened by an elliptical hole by using the Stroh-like formalism for bending theory of the anisotropic plates (Hwu, in press (a)). However, due to the restraint of Stroh-like formalism developed in our previous work (Hwu, in press (a)), our previous results for hole problems (Hsieh and Hwu, 2002a) are only valid for the pure plate bending cases, i.e., the laminates must be symmetric to avoid the coupling problems. Strictly speaking, till now no analytical solutions for the coupled stretching–bending problems of holes in composite laminates have been found in the literature. Therefore, it is necessary for us to find a simple, exact and general solution for this important problem.

Due to the two-dimensional nature, the complex variable method is good for two-dimensional problems. Among several different complex formalisms, the Stroh formalism (Stroh, 1958; Ting, 1996) has been proved to be a powerful and elegant formalism for two-dimensional linear anisotropic elasticity. Following the spirit of Stroh formalism, recently we developed a Stroh-like formalism for the pure plate bending problems of the anisotropic plates (Hwu, in press (a)) and for the coupled stretching–bending problems of the composite laminates (Hwu, in press (b)). The former formalism can also be directly applied to the symmetric composite laminates of which the bending and in-plane deformation are uncoupled, and has been successfully applied to the holes/cracks/inclusions problems subjected to out-of-plane bending moments (Hsieh and Hwu, 2002a). Now, we like to apply the latter formalism to deal with the coupled stretching–bending problems of holes in general composite laminates.

2. Stroh-like formalism for composite laminates

By combining the assumptions for lamination theory, the kinematic relations, the constitutive laws and the equilibrium equations, the governing equations for the composite laminates can be written in terms of three unknown mid-plane displacement functions u_1, u_2 and w as (Jones, 1974)

$$\begin{aligned}
&A_{11}\frac{\partial^2 u_1}{\partial x_1^2} + 2A_{16}\frac{\partial^2 u_1}{\partial x_1 \partial x_2} + A_{66}\frac{\partial^2 u_1}{\partial x_2^2} + A_{16}\frac{\partial^2 u_2}{\partial x_1^2} + (A_{12}+A_{66})\frac{\partial^2 u_2}{\partial x_1 \partial x_2} + A_{26}\frac{\partial^2 u_2}{\partial x_2^2} \\
&\quad - B_{11}\frac{\partial^3 w}{\partial x_1^3} - 3B_{16}\frac{\partial^3 w}{\partial x_1^2 \partial x_2} - (B_{12}+2B_{66})\frac{\partial^3 w}{\partial x_1 \partial x_2^2} - B_{26}\frac{\partial^3 w}{\partial x_2^3} = 0, \\
&A_{16}\frac{\partial^2 u_1}{\partial x_1^2} + (A_{12}+A_{66})\frac{\partial^2 u_1}{\partial x_1 \partial x_2} + A_{26}\frac{\partial^2 u_1}{\partial x_2^2} + A_{66}\frac{\partial^2 u_2}{\partial x_1^2} + 2A_{26}\frac{\partial^2 u_2}{\partial x_1 \partial x_2} + A_{22}\frac{\partial^2 u_2}{\partial x_2^2} \\
&\quad - B_{16}\frac{\partial^3 w}{\partial x_1^3} - (B_{12}+2B_{66})\frac{\partial^3 w}{\partial x_1^2 \partial x_2} - 3B_{26}\frac{\partial^3 w}{\partial x_1 \partial x_2^2} - B_{22}\frac{\partial^3 w}{\partial x_2^3} = 0, \\
&D_{11}\frac{\partial^4 w}{\partial x_1^4} + 4D_{16}\frac{\partial^4 w}{\partial x_1^3 \partial x_2} + 2(D_{12}+2D_{66})\frac{\partial^4 w}{\partial x_1^2 \partial x_2^2} + 4D_{26}\frac{\partial^4 w}{\partial x_1 \partial x_2^3} + D_{22}\frac{\partial^4 w}{\partial x_2^4} \\
&\quad - B_{11}\frac{\partial^3 u_1}{\partial x_1^3} - 3B_{16}\frac{\partial^3 u_1}{\partial x_1^2 \partial x_2} - (B_{12}+2B_{66})\frac{\partial^3 u_1}{\partial x_1 \partial x_2^2} - B_{26}\frac{\partial^3 u_1}{\partial x_2^3} \\
&\quad - B_{16}\frac{\partial^3 u_2}{\partial x_1^3} - (B_{12}+2B_{66})\frac{\partial^3 u_2}{\partial x_1^2 \partial x_2} - 3B_{26}\frac{\partial^3 u_2}{\partial x_1 \partial x_2^2} - B_{22}\frac{\partial^3 u_2}{\partial x_2^3} = q,
\end{aligned} \tag{1}$$

where x_1–x_2–x_3 is the commonly used Cartesian coordinate system, q is the lateral distributed load applied on laminates. A_{ij}, B_{ij} and D_{ij} are respectively, the *extensional*, *coupling* and *bending stiffnesses* which are determined by

$$A_{ij} = \sum_{k=1}^{n} (C^t_{ij})_k (h_k - h_{k-1}),$$
$$B_{ij} = \frac{1}{2} \sum_{k=1}^{n} (C^t_{ij})_k (h_k^2 - h_{k-1}^2), \tag{2}$$
$$D_{ij} = \frac{1}{3} \sum_{k=1}^{n} (C^t_{ij})_k (h_k^3 - h_{k-1}^3),$$

in which $(C^t_{ij})_k$ is the transformed stiffness of the kth lamina. h_k and h_{k-1} are, respectively, the location of the bottom and top surfaces of the kth lamina (Fig. 1).

The governing equations (1) involve both in-plane and plate bending problems, i.e., these two problems are coupled each other if the coupling stiffness B_{ij} are not equal to zero. Due to the mathematical complexity, it is really not easy to find a solution satisfying the governing equations (1) together with the complicated boundary conditions for the complicated geometrical boundaries by using the conventional method. Therefore, very few analytical solutions can be found in the literature. Based upon the two-dimensional nature of the plate bending problems and the powerful and elegant feature of the complex variable method for two-dimensional problems, recently Hwu (in press (b)) developed a Stroh-like formalism for the general composite laminates (symmetric/unsymmetric laminates). In Hwu's paper (in press (b)), two different versions of Stroh-like formalism are introduced. One is displacement-based formalism and the other is mix-based formalism. From the comparison with the Stroh formalism for two-dimensional problems, Hwu (in press (b)) observed that the displacement-based formalism is alike to the Stroh formalism in general solution, whereas the mix-based formalism is alike in the eigen-relation. Therefore, in this paper, to solve the hole problems, we will use the displacement-based formalism for the general solutions and the mix-based formalism for the eigen-relation.

According to the displacement-based formalism, the general solutions satisfying the governing equations (1) with the lateral load q neglected can be written in a compact matrix form as (Hwu, in press (b))

$$\mathbf{u}_d = 2\mathrm{Re}\{\mathbf{A}\mathbf{f}(z)\}, \qquad \boldsymbol{\phi}_d = 2\mathrm{Re}\{\mathbf{B}\mathbf{f}(z)\}, \tag{3a}$$

where

$$\mathbf{u}_d = \begin{Bmatrix} \mathbf{u} \\ \boldsymbol{\beta} \end{Bmatrix}, \qquad \boldsymbol{\phi}_d = \begin{Bmatrix} \boldsymbol{\phi} \\ \boldsymbol{\psi} \end{Bmatrix}, \tag{3b}$$

$$\mathbf{A} = [\,\mathbf{a}_1 \quad \mathbf{a}_2 \quad \mathbf{a}_3 \quad \mathbf{a}_4\,], \qquad \mathbf{B} = [\,\mathbf{b}_1 \quad \mathbf{b}_2 \quad \mathbf{b}_3 \quad \mathbf{b}_4\,], \tag{3c}$$

$$\mathbf{f}(z) = \begin{Bmatrix} f_1(z_1) \\ f_2(z_2) \\ f_3(z_3) \\ f_4(z_4) \end{Bmatrix}, \qquad z_k = x_1 + \mu_k x_2, \qquad k = 1,2,3,4, \tag{3d}$$

and

$$\mathbf{u} = \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}, \qquad \boldsymbol{\beta} = \begin{Bmatrix} \beta_1 \\ \beta_2 \end{Bmatrix} = \begin{Bmatrix} -w_{,1} \\ -w_{,2} \end{Bmatrix}, \qquad \boldsymbol{\phi} = \begin{Bmatrix} \phi_1 \\ \phi_2 \end{Bmatrix}, \qquad \boldsymbol{\psi} = \begin{Bmatrix} \psi_1 \\ \psi_2 \end{Bmatrix}, \tag{3e}$$

in which the subscript comma denotes differentiation.

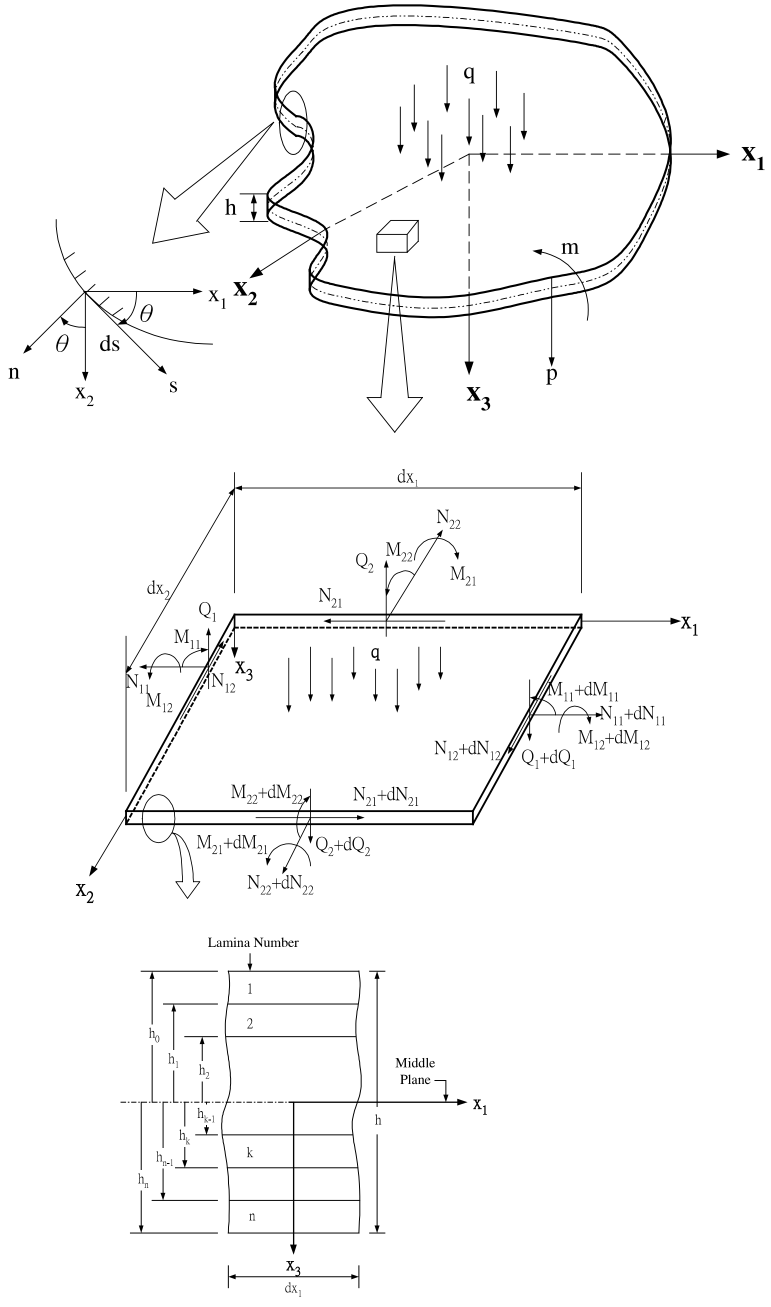


Fig. 1. Loading and geometry of a composite laminate.

In Eq. (3e), ϕ_i, $i = 1, 2$, are the stress functions related to the in-plane forces N_{ij}, and ψ_i, $i = 1, 2$, are the stress functions related to the bending moments M_{ij}, transverse shear forces Q_i and effective transverse shear forces V_i. Their relations are (Hwu, in press (b))

$$\begin{aligned}&N_{i1}=-\phi_{i,2},\qquad N_{i2}=\phi_{i,1},\\&M_{i1}=-\psi_{i,2}-\tfrac{1}{2}\lambda_{i1}\psi_{k,k},\qquad M_{i2}=\psi_{i,1}-\tfrac{1}{2}\lambda_{i2}\psi_{k,k},\\&Q_1=-\tfrac{1}{2}\psi_{k,k2},\qquad Q_2=\tfrac{1}{2}\psi_{k,k1},\\&V_1=-\psi_{2,22},\qquad V_2=\psi_{1,11},\end{aligned}\tag{4}$$

where λ_{ij} is the permutation tensor defined as $\lambda_{11}=\lambda_{22}=0$, $\lambda_{12}=-\lambda_{21}=1$.

The material eigenvalues μ_k and their associated eigenvectors $\mathbf{a}_k$ and $\mathbf{b}_k$ are determined from the following eigen-relation (Hwu, in press (b))

$$\mathbf{N}\xi=\mu\xi,\tag{5a}$$

where

$$\mathbf{N}=\mathbf{I}_t\mathbf{N}_m\mathbf{I}_t,\qquad \xi=\begin{Bmatrix}\mathbf{a}\\\mathbf{b}\end{Bmatrix},\tag{5b}$$

and

$$\mathbf{N}=\begin{bmatrix}\mathbf{N}_1&\mathbf{N}_2\\\mathbf{N}_3&\mathbf{N}_1^T\end{bmatrix},\qquad \mathbf{N}_m=\begin{bmatrix}(\mathbf{N}_m)_1&(\mathbf{N}_m)_2\\(\mathbf{N}_m)_3&(\mathbf{N}_m)_1^T\end{bmatrix},\qquad \mathbf{I}_t=\begin{bmatrix}\mathbf{I}_1&\mathbf{I}_2\\\mathbf{I}_2&\mathbf{I}_1\end{bmatrix},\tag{5c}$$

$$(\mathbf{N}_m)_1=-\mathbf{T}_m^{-1}\mathbf{R}_m^T,\qquad (\mathbf{N}_m)_2=\mathbf{T}_m^{-1},\qquad (\mathbf{N}_m)_3=\mathbf{R}_m\mathbf{T}_m^{-1}\mathbf{R}_m^T-\mathbf{Q}_m,\tag{5d}$$

$$\mathbf{I}_1=\begin{bmatrix}\mathbf{I}&\mathbf{0}\\\mathbf{0}&\mathbf{0}\end{bmatrix},\qquad \mathbf{I}_2=\begin{bmatrix}\mathbf{0}&\mathbf{0}\\\mathbf{0}&\mathbf{I}\end{bmatrix}.\tag{5e}$$

Note that the material eigenvalues μ_k have been assumed to be distinct in the general expression (3). Moreover, the four pairs of material eigenvectors $(\mathbf{a}_k,\mathbf{b}_k)$, $k=1,2,3,4$, are assumed to be those corresponding to the eigenvalues with positive imaginary parts. For the materials whose eigenvalues are repeated, a small perturbation in their values may be introduced to avoid the degenerate problems (Hwu, 1991), or a modification on the general solution can be done (Ting, 1996).

In (5b), $\mathbf{N}_m$ is the fundamental matrix of the mix-based formalism whose explicit expressions have been found in (Hsieh and Hwu, 2002b). In (5d), the three 4×4 real matrices $\mathbf{Q}_m$, $\mathbf{R}_m$ and $\mathbf{T}_m$ are defined as

$$\mathbf{Q}_m=\begin{bmatrix}\widetilde{A}_{11}&\widetilde{A}_{16}&\frac{1}{2}\widetilde{B}_{16}&\widetilde{B}_{12}\\\widetilde{A}_{16}&\widetilde{A}_{66}&\frac{1}{2}\widetilde{B}_{66}&\widetilde{B}_{62}\\\frac{1}{2}\widetilde{B}_{16}&\frac{1}{2}\widetilde{B}_{66}&-\frac{1}{4}\widetilde{D}_{66}&-\frac{1}{2}\widetilde{D}_{26}\\\widetilde{B}_{12}&\widetilde{B}_{62}&-\frac{1}{2}\widetilde{D}_{26}&-\widetilde{D}_{22}\end{bmatrix},\qquad \mathbf{R}_m=\begin{bmatrix}\widetilde{A}_{16}&\widetilde{A}_{12}&-\widetilde{B}_{11}&-\frac{1}{2}\widetilde{B}_{16}\\\widetilde{A}_{66}&\widetilde{A}_{26}&-\widetilde{B}_{61}&-\frac{1}{2}\widetilde{B}_{66}\\\frac{1}{2}\widetilde{B}_{66}&\frac{1}{2}\widetilde{B}_{26}&\frac{1}{2}\widetilde{D}_{16}&\frac{1}{4}\widetilde{D}_{66}\\\widetilde{B}_{62}&\widetilde{B}_{22}&\widetilde{D}_{12}&\frac{1}{2}\widetilde{D}_{26}\end{bmatrix},$$

$$\mathbf{T}_m=\begin{bmatrix}\widetilde{A}_{66}&\widetilde{A}_{26}&-\widetilde{B}_{61}&-\frac{1}{2}\widetilde{B}_{66}\\\widetilde{A}_{26}&\widetilde{A}_{22}&-\widetilde{B}_{21}&-\frac{1}{2}\widetilde{B}_{26}\\-\widetilde{B}_{61}&-\widetilde{B}_{21}&-\widetilde{D}_{11}&-\frac{1}{2}\widetilde{D}_{16}\\-\frac{1}{2}\widetilde{B}_{66}&-\frac{1}{2}\widetilde{B}_{26}&-\frac{1}{2}\widetilde{D}_{16}&-\frac{1}{4}\widetilde{D}_{66}\end{bmatrix},\tag{6}$$

and $\widetilde{A}_{ij},\widetilde{B}_{ij}$ and $\widetilde{D}_{ij}$ are components of the matrices $\widetilde{\boldsymbol{A}}$, $\widetilde{\boldsymbol{B}}$ and $\widetilde{\boldsymbol{D}}$ which are related to the extensional, bending and coupling stiffness matrices $\boldsymbol{A}$, $\boldsymbol{B}$ and $\boldsymbol{D}$ by

$$\widetilde{\boldsymbol{A}}=\boldsymbol{A}-\boldsymbol{B}\boldsymbol{D}^{-1}\boldsymbol{B},\qquad \widetilde{\boldsymbol{B}}=\boldsymbol{B}\boldsymbol{D}^{-1},\qquad \widetilde{\boldsymbol{D}}=\boldsymbol{D}^{-1}.\tag{7}$$

Note that in the above matrix expressions, the symbols $\boldsymbol{A}$ and $\boldsymbol{B}$ have different representations from the eigenvector matrices $\mathbf{A}$ and $\mathbf{B}$ defined in (3c). The former is the traditional notation used in the community of mechanics of composite materials, while the latter is the notation generally used in the community of anisotropic elasticity. To let the readers from both communities see clearly what we express in this paper, we just use the italic bold-faced fonts to denote the extensional and coupling matrices and the roman bold-faced fonts to denote the material eigenvector matrices.

As we said previously, in eigen-relation the mix-based formalism is more alike to the Stroh formalism. Therefore, it is easier to find the explicit expressions for $\mathbf{N}_{\rm m}$ by following the steps employed in the Stroh formalism (Hsieh and Hwu, 2002b). After getting $\mathbf{N}_{\rm m}$, we can then use the relation provided in $(5b)_1$ to obtain the explicit expression of the fundamental matrix $\mathbf{N}$ for the displacement-based formalism which are now listed in Appendix A.

Since the mathematical form of the eigen-relation (5a) and the characteristics of fundamental matrix $\mathbf{N}$ $(5c)_1$ are exactly the same as the Stroh formalism for two-dimensional problems (Ting, 1996), all the identities derived through these relations for Stroh formalism should automatically be transferred to the present Stroh-like formalism. Hence, without any further derivation, we may now write down the following identities which have been proved in Stroh formalism for two-dimensional problems (Ting, 1996),

$$\begin{aligned}
\mathbf{A}\langle\mu_k\rangle\mathbf{A}^{-1} &= \mathbf{N}_1 + \mathbf{N}_2\mathbf{S}^{\rm T}\mathbf{H}^{-1} + {\rm i}\mathbf{N}_2\mathbf{H}^{-1},\\
\mathbf{B}\langle\mu_k\rangle\mathbf{A}^{-1} &= \mathbf{N}_3 + \mathbf{N}_1^{\rm T}\mathbf{S}^{\rm T}\mathbf{H}^{-1} + {\rm i}\mathbf{N}_1^{\rm T}\mathbf{H}^{-1},\\
\mathbf{B}\langle\mu_k\rangle\mathbf{B}^{-1} &= \mathbf{N}_1^{\rm T} - \mathbf{N}_3\mathbf{S}\mathbf{L}^{-1} - {\rm i}\mathbf{N}_3\mathbf{L}^{-1},
\end{aligned} \tag{8}$$

where the angular bracket $\langle\ \rangle$ stands for the diagonal matrix; $\mathbf{N}_1$, $\mathbf{N}_2$, $\mathbf{N}_3$ are the sub-matrix of the fundamental matrix $\mathbf{N}$ whose explicit expressions are shown in Appendix A, and $\mathbf{S}$, $\mathbf{H}$ and $\mathbf{L}$, generally called Barnett–Lothe tensors in two-dimensional problems, are three real matrices defined as

$$\mathbf{S} = {\rm i}(2\mathbf{A}\mathbf{B}^{\rm T} - \mathbf{I}), \qquad \mathbf{H} = 2{\rm i}\mathbf{A}\mathbf{A}^{\rm T}, \qquad \mathbf{L} = -2{\rm i}\mathbf{B}\mathbf{B}^{\rm T}. \tag{9}$$

Like the generalized eigen-relation for the two-dimensional problems (Ting, 1996), the eigen-relation (5) can be generalized as

$$\mathbf{N}(\omega)\xi = \mu(\omega)\xi, \tag{10a}$$

where

$$\mathbf{N}(\omega) = \mathbf{I}_t\mathbf{N}_{\rm m}(\omega)\mathbf{I}_t, \qquad \xi = \begin{Bmatrix}\mathbf{a}\\ \mathbf{b}\end{Bmatrix}, \tag{10b}$$

$$\mathbf{N}(\omega) = \begin{bmatrix}\mathbf{N}_1(\omega) & \mathbf{N}_2(\omega)\\ \mathbf{N}_3(\omega) & \mathbf{N}_1^{\rm T}(\omega)\end{bmatrix}, \qquad \mathbf{N}_{\rm m}(\omega) = \begin{bmatrix}(\mathbf{N}_{\rm m}(\omega))_1 & (\mathbf{N}_{\rm m}(\omega))_2\\ (\mathbf{N}_{\rm m}(\omega))_3 & (\mathbf{N}_{\rm m}(\omega))_1^{\rm T}\end{bmatrix}, \tag{10c}$$

$$\begin{aligned}
&(\mathbf{N}_{\rm m}(\omega))_1 = -\mathbf{T}_{\rm m}^{-1}(\omega)\mathbf{R}_{\rm m}^{\rm T}(\omega), \qquad (\mathbf{N}_{\rm m}(\omega))_2 = \mathbf{T}_{\rm m}^{-1}(\omega),\\
&(\mathbf{N}_{\rm m}(\omega))_3 = \mathbf{R}_{\rm m}(\omega)\mathbf{T}_{\rm m}^{-1}(\omega)\mathbf{R}_{\rm m}^{\rm T}(\omega) - \mathbf{Q}_{\rm m}(\omega),
\end{aligned} \tag{10d}$$

and

$$\mu(\omega) = \frac{-\sin\omega + \mu\cos\omega}{\cos\omega + \mu\sin\omega}. \tag{10e}$$

In (10d), $\mathbf{Q}_{\rm m}(\omega)$, $\mathbf{R}_{\rm m}(\omega)$ and $\mathbf{T}_{\rm m}(\omega)$ are related to the matrices $\mathbf{Q}_{\rm m}$, $\mathbf{R}_{\rm m}$ and $\mathbf{T}_{\rm m}$ defined in (6) by

$$\begin{aligned}
\mathbf{Q}_{\rm m}(\omega) &= \mathbf{Q}_{\rm m}\cos^2\omega + (\mathbf{R}_{\rm m} + \mathbf{R}_{\rm m}^{\rm T})\sin\omega\cos\omega + \mathbf{T}_{\rm m}\sin^2\omega,\\
\mathbf{R}_{\rm m}(\omega) &= \mathbf{R}_{\rm m}\cos^2\omega + (\mathbf{T}_{\rm m} - \mathbf{Q}_{\rm m})\sin\omega\cos\omega - \mathbf{R}_{\rm m}^{\rm T}\sin^2\omega,\\
\mathbf{T}_{\rm m}(\omega) &= \mathbf{T}_{\rm m}\cos^2\omega - (\mathbf{R}_{\rm m} + \mathbf{R}_{\rm m}^{\rm T})\sin\omega\cos\omega + \mathbf{Q}_{\rm m}\sin^2\omega,
\end{aligned} \tag{11}$$

in which ω denotes the angle between the transformed and original coordinates such as the angle θ shown in Fig. 1.

Using (10) and following the same procedures for the derivation of the identities (8), we can obtain the useful generalized identities as

$$\begin{aligned}\mathbf{A}\langle\mu_k(\omega)\rangle\mathbf{A}^{-1} &= \mathbf{E}_1(\omega) + \mathrm{i}\mathbf{F}_1(\omega),\\ \mathbf{B}\langle\mu_k(\omega)\rangle\mathbf{A}^{-1} &= \mathbf{E}_2(\omega) + \mathrm{i}\mathbf{F}_2(\omega),\\ \mathbf{B}\langle\mu_k(\omega)\rangle\mathbf{B}^{-1} &= \mathbf{E}_3(\omega) + \mathrm{i}\mathbf{F}_3(\omega),\end{aligned} \tag{12a}$$

where

$$\begin{aligned}\mathbf{E}_1(\omega) &= \mathbf{N}_1(\omega) + \mathbf{N}_2(\omega)\mathbf{S}^{\mathrm{T}}\mathbf{H}^{-1}, &\quad \mathbf{F}_1(\omega) &= \mathbf{N}_2(\omega)\mathbf{H}^{-1},\\ \mathbf{E}_2(\omega) &= \mathbf{N}_3(\omega) + \mathbf{N}_1^{\mathrm{T}}(\omega)\mathbf{S}^{\mathrm{T}}\mathbf{H}^{-1}, &\quad \mathbf{F}_2(\omega) &= \mathbf{N}_1^{\mathrm{T}}(\omega)\mathbf{H}^{-1},\\ \mathbf{E}_3(\omega) &= \mathbf{N}_1^{\mathrm{T}}(\omega) - \mathbf{N}_3(\omega)\mathbf{S}\mathbf{L}^{-1}, &\quad \mathbf{F}_3(\omega) &= -\mathbf{N}_3(\omega)\mathbf{L}^{-1}.\end{aligned} \tag{12b}$$

3. Forces, moments and transverse shear forces

In general, one is usually interested in the forces, moments and shear forces around the hole boundary since most of the critical stress occurs along the hole boundary. To find the forces, moments and transverse shear forces around the hole boundary, the conventional way is to calculate the forces (N_{11}, N_{22}, N_{12}), moments (M_{11}, M_{22}, M_{12}) and transverse shear forces (Q_1, Q_2) or effective transverse shear forces (V_1, V_2) by using their relations with the deflection w, and then use the transformation law to find their values in the normal and tangent coordinate system, i.e. N_n, N_s, N_{ns}, M_n, M_s, M_{ns}, Q_n, Q_s, V_n, V_s. Since in our Stroh-like formalism for general composite laminates, the final results are expressed in terms of the displacement vector $\mathbf{u}_d$ and the stress function vector $\boldsymbol{\phi}_d$, it is hoped that the forces, moments and the transverse shear forces can be found directly from $\boldsymbol{\phi}_d$ instead of using the conventional way.

Like the surface traction for two-dimensional problems, we may now define the surface forces $\mathbf{t}_n$ and surface moments $\mathbf{m}_n$ along the hole boundary with normal $\mathbf{n}$ as

$$(\mathbf{t}_n)_i = N_{ij}n_j, \qquad (\mathbf{m}_n)_i = M_{ij}n_j, \tag{13a}$$

and the surface forces $\mathbf{t}_s$ and surface moments $\mathbf{m}_s$ along the surface perpendicular to the hole boundary with normal $\mathbf{s}$ as

$$(\mathbf{t}_s)_i = N_{ij}s_j, \qquad (\mathbf{m}_s)_i = M_{ij}s_j, \tag{13b}$$

in which

$$n_1 = -s_2 = -\sin\theta = -\frac{\partial x_2}{\partial s} = \frac{\partial x_1}{\partial n}, \qquad n_2 = s_1 = \cos\theta = \frac{\partial x_1}{\partial s} = \frac{\partial x_2}{\partial n}, \tag{13c}$$

where θ denotes the angle from the positive x_1-axis to the direction s in clockwise direction (see Fig. 1).

Substituting (4) into (13a) and (13b), we get

$$\begin{aligned}\mathbf{t}_n &= \boldsymbol{\phi}_{,s}, &\qquad \mathbf{m}_n &= \boldsymbol{\psi}_{,s} - \eta\mathbf{s},\\ \mathbf{t}_s &= -\boldsymbol{\phi}_{,n}, &\qquad \mathbf{m}_s &= -\boldsymbol{\psi}_{,n} + \eta\mathbf{n},\end{aligned} \tag{14a}$$

where

$$\eta = \frac{1}{2}\left(\frac{\partial\psi_1}{\partial x_1} + \frac{\partial\psi_2}{\partial x_2}\right). \tag{14b}$$

The resultant forces and moments in s–n coordinates can therefore be calculated by

$$\begin{aligned} &N_n = \mathbf{n}^T\mathbf{t}_n = \mathbf{n}^T\boldsymbol{\phi}_{,s}, \qquad N_{ns} = \mathbf{s}^T\mathbf{t}_n = \mathbf{s}^T\boldsymbol{\phi}_{,s}, \\ &N_s = \mathbf{s}^T\mathbf{t}_s = -\mathbf{s}^T\boldsymbol{\phi}_{,n}, \qquad N_{sn} = \mathbf{n}^T\mathbf{t}_s = -\mathbf{n}^T\boldsymbol{\phi}_{,n} = N_{ns}, \\ &M_n = \mathbf{n}^T\mathbf{m}_n = \mathbf{n}^T\boldsymbol{\psi}_{,s}, \qquad M_{ns} = \mathbf{s}^T\mathbf{m}_n = \mathbf{s}^T\boldsymbol{\psi}_{,s} - \eta, \\ &M_s = \mathbf{s}^T\mathbf{m}_s = -\mathbf{s}^T\boldsymbol{\psi}_{,n}, \qquad M_{sn} = \mathbf{n}^T\mathbf{m}_s = -\mathbf{n}^T\boldsymbol{\psi}_{,n} + \eta = M_{ns}. \end{aligned} \tag{15}$$

By the equality of M_{ns} and M_{sn}, we have

$$\eta = \tfrac{1}{2}(\mathbf{s}^T\boldsymbol{\psi}_{,s} + \mathbf{n}^T\boldsymbol{\psi}_{,n}). \tag{16}$$

The transverse shear forces and effective transverse shear forces in s–n coordinates can be obtained by utilizing the transformation laws, $(4)_{5–8}$, (14) and (15) which can be expressed as

$$Q_n = \eta_{,s}, \qquad Q_s = -\eta_{,n}, \tag{17a}$$

$$V_n = Q_n + \frac{\partial M_{ns}}{\partial s} = (\mathbf{s}^T\boldsymbol{\psi}_{,s})_{,s}, \qquad V_s = Q_s + \frac{\partial M_{ns}}{\partial n} = -(\mathbf{n}^T\boldsymbol{\psi}_{,n})_{,n}, \tag{17b}$$

From the above equations, we see that all the forces and moments in s–n coordinates have simple relations with the stress function vector $\boldsymbol{\phi}_d$ $(= (\boldsymbol{\phi}, \boldsymbol{\psi})^T)$.

4. Elliptical holes

Consider an unbounded composite laminate with an elliptical hole subjected to in-plane forces $N_{11} = N_{11}^\infty$, $N_{22} = N_{22}^\infty$, $N_{12} = N_{12}^\infty$ and out-of-plane bending moments $M_{11} = M_{11}^\infty$, $M_{22} = M_{22}^\infty$, $M_{12} = M_{12}^\infty$ at infinity (Fig. 2). There is no load around the edge of the elliptical hole. The contour of the elliptical hole is represented by

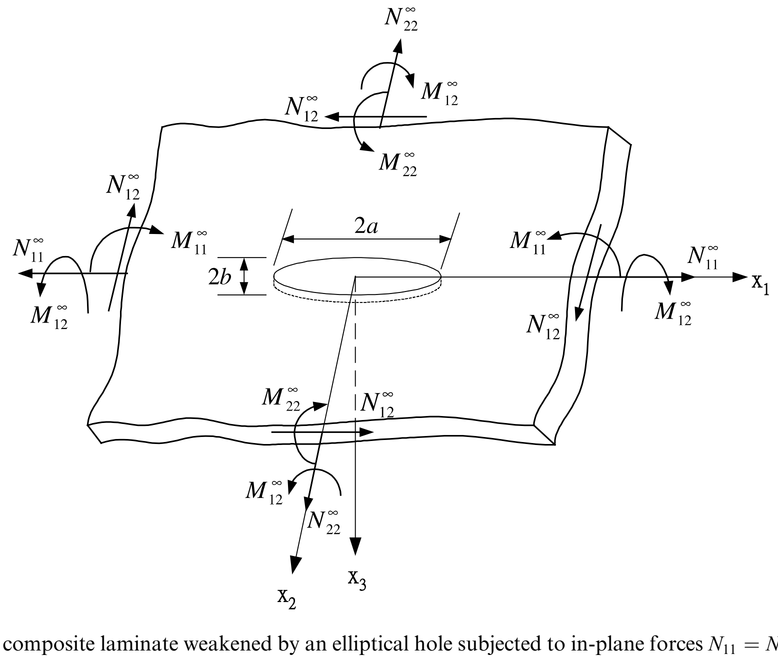


Fig. 2. An unsymmetric composite laminate weakened by an elliptical hole subjected to in-plane forces $N_{11} = N_{11}^\infty$, $N_{22} = N_{22}^\infty$, $N_{12} = N_{12}^\infty$ and out-of-plane bending moments $M_{11} = M_{11}^\infty$, $M_{22} = M_{22}^\infty$, $M_{12} = M_{12}^\infty$.

$$x_1 = a\cos\varphi, \qquad x_2 = b\sin\varphi, \tag{18}$$

where $2a$, $2b$ are the major and minor axes of the ellipse and φ is a real parameter. The boundary conditions of this problem can be expressed as

$$\begin{matrix} N_{11} = N_{11}^{\infty}, & N_{22} = N_{22}^{\infty}, & N_{12} = N_{12}^{\infty}, \\ M_{11} = M_{11}^{\infty}, & M_{22} = M_{22}^{\infty}, & M_{12} = M_{12}^{\infty}, \end{matrix} \quad \text{at infinity,} \tag{19a}$$

$$N_n = N_{ns} = 0, \qquad M_n = V_n = 0 \quad \text{along the hole boundary.} \tag{19b}$$

By using the displacement function vector $\mathbf{u}_d$ and stress function vector $\boldsymbol{\phi}_d$ introduced in (3b) with their definitions given in (3e) and the relations derived in (4), (15)–(17), the boundary condition (19) can be expressed in terms of the stress function as

$$\boldsymbol{\phi}_d = \boldsymbol{\phi}_d^{\infty} = x_1\mathbf{m}_2^{\infty} - x_2\mathbf{m}_1^{\infty} \quad \text{at infinity,} \tag{20a}$$

$$\boldsymbol{\phi}_d = \mathbf{0} \quad \text{along the hole boundary,} \tag{20b}$$

where

$$\mathbf{m}_1^{\infty} = \begin{Bmatrix} N_{11}^{\infty} \\ N_{12}^{\infty} \\ M_{11}^{\infty} \\ M_{12}^{\infty} \end{Bmatrix}, \qquad \mathbf{m}_2^{\infty} = \begin{Bmatrix} N_{12}^{\infty} \\ N_{22}^{\infty} \\ M_{12}^{\infty} \\ M_{22}^{\infty} \end{Bmatrix}. \tag{21}$$

The displacement vector $\mathbf{u}_d^{\infty}$ corresponding to $\boldsymbol{\phi}_d^{\infty}$ for a homogeneous composite laminate may be obtained from the constitutive laws (Jones, 1974), i.e.,

$$\begin{Bmatrix} \boldsymbol{N} \\ \boldsymbol{M} \end{Bmatrix} = \begin{bmatrix} \boldsymbol{A} & \boldsymbol{B} \\ \boldsymbol{B} & \boldsymbol{D} \end{bmatrix} \begin{Bmatrix} \boldsymbol{\epsilon}_0 \\ \boldsymbol{\kappa} \end{Bmatrix} \Rightarrow \begin{Bmatrix} \boldsymbol{\epsilon}_0 \\ \boldsymbol{\kappa} \end{Bmatrix} = \begin{bmatrix} \boldsymbol{A}^* & \boldsymbol{B}^* \\ \boldsymbol{B}^* & \boldsymbol{D}^* \end{bmatrix} \begin{Bmatrix} \boldsymbol{N} \\ \boldsymbol{M} \end{Bmatrix}, \tag{22a}$$

where

$$\boldsymbol{N} = \begin{Bmatrix} N_{11} \\ N_{22} \\ N_{12} \end{Bmatrix}, \quad \boldsymbol{M} = \begin{Bmatrix} M_{11} \\ M_{22} \\ M_{12} \end{Bmatrix}, \quad \boldsymbol{\epsilon}_0 = \begin{Bmatrix} \varepsilon_{11}^0 \\ \varepsilon_{22}^0 \\ \gamma_{12}^0 \end{Bmatrix}, \quad \boldsymbol{\kappa} = \begin{Bmatrix} \kappa_{11} \\ \kappa_{22} \\ \kappa_{12} \end{Bmatrix}, \tag{22b}$$

and $\boldsymbol{A}^*$, $\boldsymbol{B}^*$, $\boldsymbol{D}^*$ are related to the extensional, bending and coupling stiffness matrices $\boldsymbol{A}$, $\boldsymbol{B}$ and $\boldsymbol{D}$ by

$$\boldsymbol{A}^* = \boldsymbol{A}^{-1} + \boldsymbol{A}^{-1}\boldsymbol{B}\boldsymbol{D}^*\boldsymbol{B}\boldsymbol{A}^{-1}, \qquad \boldsymbol{B}^* = -\boldsymbol{A}^{-1}\boldsymbol{B}\boldsymbol{D}^*, \qquad \boldsymbol{D}^* = (\boldsymbol{D} - \boldsymbol{B}\boldsymbol{A}^{-1}\boldsymbol{B})^{-1}. \tag{23}$$

From (22a), we get the following relation

$$\begin{Bmatrix} \mathbf{m}_1^{\infty} \\ \mathbf{m}_2^{\infty} \end{Bmatrix} = \begin{bmatrix} \mathbf{Q} & \mathbf{R} \\ \mathbf{R}^{\mathrm{T}} & \mathbf{T} \end{bmatrix} \begin{Bmatrix} \mathbf{d}_1^{\infty} \\ \mathbf{d}_2^{\infty} \end{Bmatrix} \Rightarrow \begin{Bmatrix} \mathbf{d}_1^{\infty} \\ \mathbf{d}_2^{\infty} \end{Bmatrix} = \begin{bmatrix} \mathbf{Q}^* & \mathbf{R}^* \\ \mathbf{R}^{*\mathrm{T}} & \mathbf{T}^* \end{bmatrix} \begin{Bmatrix} \mathbf{m}_1^{\infty} \\ \mathbf{m}_2^{\infty} \end{Bmatrix}, \tag{24a}$$

where $\mathbf{m}_1^{\infty}$, $\mathbf{m}_2^{\infty}$ are defined in (21) and

$$\mathbf{d}_1^{\infty} = \begin{Bmatrix} \varepsilon_{11}^{\infty} \\ \gamma_{12}^{\infty}/2 \\ \kappa_{11}^{\infty} \\ \kappa_{12}^{\infty}/2 \end{Bmatrix}, \qquad \mathbf{d}_2^{\infty} = \begin{Bmatrix} \gamma_{12}^{\infty}/2 \\ \varepsilon_{22}^{\infty} \\ \kappa_{12}^{\infty}/2 \\ \kappa_{22}^{\infty} \end{Bmatrix}. \tag{24b}$$

The three real matrices $\mathbf{Q}$, $\mathbf{R}$ and $\mathbf{T}$ which are the counterpart of $\mathbf{Q}_{\mathrm{m}}$, $\mathbf{R}_{\mathrm{m}}$ and $\mathbf{T}_{\mathrm{m}}$ of the mix-based formalism given in (6), are defined as

$$\mathbf{Q} = \begin{bmatrix} A_{11} & A_{16} & B_{11} & B_{16} \\ A_{61} & A_{66} & B_{61} & B_{66} \\ B_{11} & B_{16} & D_{11} & D_{16} \\ B_{61} & B_{66} & D_{61} & D_{66} \end{bmatrix}, \qquad \mathbf{R} = \begin{bmatrix} A_{16} & A_{12} & B_{16} & B_{12} \\ A_{66} & A_{62} & B_{66} & B_{62} \\ B_{16} & B_{12} & D_{16} & D_{12} \\ B_{66} & B_{62} & D_{66} & D_{62} \end{bmatrix},$$

$$\mathbf{T} = \begin{bmatrix} A_{66} & A_{62} & B_{66} & B_{62} \\ A_{26} & A_{22} & B_{26} & B_{22} \\ B_{66} & B_{62} & D_{66} & D_{62} \\ B_{26} & B_{22} & D_{26} & D_{22} \end{bmatrix}. \tag{25a}$$

The pseudo-inverse $\mathbf{Q}^*$, $\mathbf{R}^*$ and $\mathbf{T}^*$ are then defined by

$$\mathbf{Q}^* = \begin{bmatrix} A^*_{11} & \frac{1}{2}A^*_{16} & B^*_{11} & \frac{1}{2}B^*_{16} \\ \frac{1}{2}A^*_{16} & \frac{1}{4}A^*_{66} & \frac{1}{2}B^*_{61} & \frac{1}{4}B^*_{66} \\ B^*_{11} & \frac{1}{2}B^*_{61} & D^*_{11} & \frac{1}{2}D^*_{16} \\ \frac{1}{2}B^*_{16} & \frac{1}{4}B^*_{66} & \frac{1}{2}D^*_{16} & \frac{1}{4}D^*_{66} \end{bmatrix}, \qquad \mathbf{R}^* = \begin{bmatrix} \frac{1}{2}A^*_{16} & A^*_{12} & \frac{1}{2}B^*_{16} & B^*_{12} \\ \frac{1}{4}A^*_{66} & \frac{1}{2}A^*_{26} & \frac{1}{4}B^*_{66} & \frac{1}{2}B^*_{62} \\ \frac{1}{2}B^*_{61} & B^*_{21} & \frac{1}{2}D^*_{16} & D^*_{12} \\ \frac{1}{4}B^*_{66} & \frac{1}{2}B^*_{26} & \frac{1}{4}D^*_{66} & \frac{1}{2}D^*_{26} \end{bmatrix},$$

$$\mathbf{T}^* = \begin{bmatrix} \frac{1}{4}A^*_{66} & \frac{1}{2}A^*_{26} & \frac{1}{4}B^*_{66} & \frac{1}{2}B^*_{62} \\ \frac{1}{2}A^*_{26} & A^*_{22} & \frac{1}{2}B^*_{26} & B^*_{22} \\ \frac{1}{4}B^*_{66} & \frac{1}{2}B^*_{26} & \frac{1}{4}D^*_{66} & \frac{1}{2}D^*_{26} \\ \frac{1}{2}B^*_{62} & B^*_{22} & \frac{1}{2}D^*_{26} & D^*_{22} \end{bmatrix}, \tag{25b}$$

where A^*_{ij}, B^*_{ij} and D^*_{ij} are components of the matrices $\boldsymbol{A}^*$, $\boldsymbol{B}^*$ and $\boldsymbol{D}^*$ given in (23). Here we use the name "pseudo-inverse" just because the actual inverse of $\begin{bmatrix} \mathbf{Q} & \mathbf{R} \\ \mathbf{R}^{\mathrm{T}} & \mathbf{T} \end{bmatrix}$ does not exist since $(\mathbf{m}_1^\infty)_2 = (\mathbf{m}_2^\infty)_1 = N_{12}^\infty$, $(\mathbf{d}_1^\infty)_2 = (\mathbf{d}_2^\infty)_1 = \gamma_{12}^\infty/2$ in (24a) will make some identical rows and columns in the matrix $\begin{bmatrix} \mathbf{Q} & \mathbf{R} \\ \mathbf{R}^{\mathrm{T}} & \mathbf{T} \end{bmatrix}$, which will then be singular. By deleting the duplicated rows and columns in (24a) and adding them after inversion, we can then get the pseudo-inverse $\mathbf{Q}^*$, $\mathbf{R}^*$and $\mathbf{T}^*$.

Integration of the second equation of (24a) with respect to x_1 and x_2, respectively, now leads to the results of $\mathbf{u}_d^\infty$ as

$$\mathbf{u}_d^\infty = x_1\mathbf{d}_1^\infty + x_2\mathbf{d}_2^\infty. \tag{26}$$

In order to satisfy the boundary condition at infinity as shown in (20a), the displacement vector $\mathbf{u}_d^\infty$ and the stress function vector $\boldsymbol{\phi}_d^\infty$ are added to the general solution (3a). Whereas for the satisfaction of the hole boundary condition, by referring to the solutions of the corresponding two-dimensional problems (Hwu and Yen, 1992; Hwu, 1992; Hsieh and Hwu, 2002a), we choose

$$\mathbf{f}(z) = \langle \varsigma_k^{-1} \rangle \mathbf{k}, \qquad \varsigma_k = \frac{z_k + \sqrt{z_k^2 - a^2 - \mu_k^2 b^2}}{a - \mathrm{i}\mu_k b}, \tag{27}$$

where $\mathbf{k}$ is the unknown coefficient to be determined through the satisfaction of the hole boundary condition. Therefore, the solution for the present problem can be expressed as

$$\mathbf{u}_d = \mathbf{u}_d^\infty + 2\mathrm{Re}\{\mathbf{A}\langle \varsigma_k^{-1} \rangle \mathbf{k}\}, \tag{28a}$$

$$\boldsymbol{\phi}_d = \boldsymbol{\phi}_d^\infty + 2\mathrm{Re}\{\mathbf{B}\langle \varsigma_k^{-1} \rangle \mathbf{k}\}, \tag{28b}$$

where $\mathbf{u}_d^\infty$, $\boldsymbol{\phi}_d^\infty$ are given in (26) and (20a), respectively.

From the knowledge of (Hwu and Yen, 1992; Hwu, 1992; Hsieh and Hwu, 2002a), we know that

$$\varsigma_k = e^{-i\varphi} \quad \text{along the hole boundary.} \tag{29}$$

Substituting (28) and (29) into the boundary condition (20b), we obtain

$$\mathbf{k} = -\tfrac{1}{2}\mathbf{B}^{-1}(a\mathbf{m}_2^\infty - ib\mathbf{m}_1^\infty). \tag{30}$$

The explicit solutions can therefore be expressed as

$$\begin{aligned} \mathbf{u}_d &= \mathbf{u}_d^\infty - \mathrm{Re}\{\mathbf{A}\langle\varsigma_k^{-1}\rangle\mathbf{B}^{-1}(a\mathbf{m}_2^\infty - ib\mathbf{m}_1^\infty)\}, \\ \boldsymbol{\phi}_d &= \boldsymbol{\phi}_d^\infty - \mathrm{Re}\{\mathbf{B}\langle\varsigma_k^{-1}\rangle\mathbf{B}^{-1}(a\mathbf{m}_2^\infty - ib\mathbf{m}_1^\infty)\}, \end{aligned} \tag{31}$$

which has exactly the same form as those of the corresponding two-dimensional problems (Hwu, 1992) and pure bending problems (Hsieh and Hwu, 2002a). The only difference is that the symbols like $\mathbf{u}, \boldsymbol{\phi}, \mathbf{A}, \mathbf{B}, \ldots,$ etc. have different dimensions and different contents for different types of problems.

4.1. Forces and moments along the hole boundary

According to the relations obtained in (15)–(17) we know that the calculation of the forces and moments relies upon the calculation of the differentials $\boldsymbol{\phi}_{d,s}$ and $\boldsymbol{\phi}_{d,n}$. Again, because our final solutions for $\boldsymbol{\phi}_d$ obtained in (31) have exactly the same form as those of the corresponding two-dimensional problems, just by copying our previous corresponding results (Hwu, 1992; Hsieh and Hwu, 2002a) without any further detailed derivation, we may get

$$\boldsymbol{\phi}_{d,s} = 0, \tag{32a}$$

$$\boldsymbol{\phi}_{d,n} = -\cos\theta\left[\mathbf{m}_1^\infty + \mathbf{E}_3(\theta)\mathbf{m}_2^\infty + \frac{b}{a}\mathbf{F}_3(\theta)\mathbf{m}_1^\infty\right] - \sin\theta\left[\mathbf{m}_2^\infty - \mathbf{E}_3(\theta)\mathbf{m}_1^\infty + \frac{a}{b}\mathbf{F}_3(\theta)\mathbf{m}_2^\infty\right], \tag{32b}$$

where $\mathbf{E}_3(\theta)$ and $\mathbf{F}_3(\theta)$ are defined in $(12b)_3$, which are composed of the fundamental real matrices $\mathbf{N}_i(\theta)$, $\mathbf{S}$, $\mathbf{H}$, and $\mathbf{L}$. If one does not feel comfortable about directly copying the previous corresponding results, derivation can be done with the help of the references (Hwu, 1992; Hsieh and Hwu, 2002a) and the identities given in (8) and (12). Substituting (32) into (15), we obtain the forces and moments around the hole boundary as

$$\begin{aligned} N_n &= N_{ns} = M_n = 0, \\ N_s &= \cos\theta\left\{\hat{\sigma}_1^{(0)} - c\hat{\sigma}_1^{(3)} + \hat{\sigma}_2^{(1)}\right\} + \sin\theta\left\{\hat{\sigma}_2^{(0)} - \frac{1}{c}\hat{\sigma}_2^{(3)} - \hat{\sigma}_1^{(1)}\right\}, \\ M_s &= \cos\theta\left\{\sigma_1^{(0)} - c\sigma_1^{(3)} + \sigma_2^{(1)}\right\} + \sin\theta\left\{\sigma_2^{(0)} - \frac{1}{c}\sigma_2^{(3)} - \sigma_1^{(1)}\right\}, \\ M_{ns} &= \cos\theta\left\{\tilde{\sigma}_1^{(0)} - c\tilde{\sigma}_1^{(3)} + \tilde{\sigma}_2^{(1)}\right\} + \sin\theta\left\{\tilde{\sigma}_2^{(0)} - \frac{1}{c}\tilde{\sigma}_2^{(3)} - \tilde{\sigma}_1^{(1)}\right\}, \end{aligned} \tag{33a}$$

where

$$\begin{aligned}
&c = b/a,\\
&\hat{\sigma}_i^{(0)} = \mathbf{s}^{\mathrm{T}}(\theta)\hat{\mathbf{I}}_n\mathbf{m}_i^{\infty}, \qquad \hat{\sigma}_i^{(1)} = \mathbf{s}^{\mathrm{T}}(\theta)\hat{\mathbf{I}}_n[\mathbf{N}_1^{\mathrm{T}}(\theta) - \mathbf{N}_3(\theta)\mathbf{S}\mathbf{L}^{-1}]\mathbf{m}_i^{\infty},\\
&\hat{\sigma}_i^{(3)} = \mathbf{s}^{\mathrm{T}}(\theta)\hat{\mathbf{I}}_n\mathbf{N}_3(\theta)\mathbf{L}^{-1}\mathbf{m}_i^{\infty},\\
&\sigma_i^{(0)} = \mathbf{s}^{\mathrm{T}}(\theta)\hat{\mathbf{I}}_{\mathrm{m}}\mathbf{m}_i^{\infty}, \qquad \sigma_i^{(1)} = \mathbf{s}^{\mathrm{T}}(\theta)\hat{\mathbf{I}}_{\mathrm{m}}[\mathbf{N}_1^{\mathrm{T}}(\theta) - \mathbf{N}_3(\theta)\mathbf{S}\mathbf{L}^{-1}]\mathbf{m}_i^{\infty},\\
&\sigma_i^{(3)} = \mathbf{s}^{\mathrm{T}}(\theta)\hat{\mathbf{I}}_{\mathrm{m}}\mathbf{N}_3(\theta)\mathbf{L}^{-1}\mathbf{m}_i^{\infty},\\
&\tilde{\sigma}_i^{(0)} = \frac{1}{2}\mathbf{n}^{\mathrm{T}}(\theta)\hat{\mathbf{I}}_{\mathrm{m}}\mathbf{m}_i^{\infty}, \qquad \tilde{\sigma}_i^{(1)} = \frac{1}{2}\mathbf{n}^{\mathrm{T}}(\theta)\hat{\mathbf{I}}_{\mathrm{m}}[\mathbf{N}_1^{\mathrm{T}}(\theta) - \mathbf{N}_3(\theta)\mathbf{S}\mathbf{L}^{-1}]\mathbf{m}_i^{\infty},\\
&\tilde{\sigma}_i^{(3)} = \frac{1}{2}\mathbf{n}^{\mathrm{T}}(\theta)\hat{\mathbf{I}}_{\mathrm{m}}\mathbf{N}_3(\theta)\mathbf{L}^{-1}\mathbf{m}_i^{\infty}, \quad i = 1,2,\\
&\hat{\mathbf{I}}_n = [\mathbf{I} \quad \mathbf{0}] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}, \qquad \hat{\mathbf{I}}_{\mathrm{m}} = [\mathbf{0} \quad \mathbf{I}] = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.
\end{aligned} \tag{33b}$$

5. Numerical examples

To show that the explicit solutions obtained in (33) for the forces and moments around the hole boundary are exact, several examples are illustrated in this section. Not only the resultant forces and moments around the hole boundary are presented but also the discussion for the coupling effect of the laminates and the shape effect of the holes are shown in this section. All the examples consider an unbounded laminate composed of different combinations of graphite–epoxy fiber-reinforced composite laminae. The laminate contains a through-thickness elliptical hole. Each lamina thickness is 1 mm, and the material properties of the graphite/epoxy are

$$E_1 = 138 \text{ GPa}, \qquad E_2 = 9 \text{ GPa}, \qquad G_{12} = 6.9 \text{ GPa}, \qquad v_{12} = 0.3,$$

where E_1 and E_2 are the Young's moduli in x_1 and x_2 directions, respectively; G_{12} is the shear modulus in the x_1x_2 plane; v_{12} is the major Poisson's ratio, and has the following relation with the minor Poisson's ratio v_{21} as

$$v_{21}E_1 = v_{12}E_2. \tag{34}$$

Example 1. A circular hole in a unidirectional laminate

Consider a [+45]$_3$ unidirectional laminate subjected to out-of-plane bending moment $M_{11}^{\infty} = \hat{m}$. Since the laminate is composed of the same laminae, the laminate is also a symmetric laminate. According to the generalized plane stress theory, the stiffness matrix $\mathbf{C}$ referred to the principle material directions can be calculated as

$$\mathbf{C} = \begin{bmatrix} \dfrac{E_1}{1 - v_{12}v_{21}} & \dfrac{v_{21}E_1}{1 - v_{12}v_{21}} & 0 \\ \dfrac{v_{21}E_1}{1 - v_{12}v_{21}} & \dfrac{E_2}{1 - v_{12}v_{21}} & 0 \\ 0 & 0 & G_{12} \end{bmatrix} = \begin{bmatrix} 138.8 & 2.72 & 0 \\ 2.72 & 9.05 & 0 \\ 0 & 0 & 6.9 \end{bmatrix} \text{ GPa}. \tag{35}$$

The transformed stiffness matrix $\mathbf{C}^{\mathrm{t}}$ of each lamina can then be calculated by the 4th order transformation law as

$$\mathbf{C}^{\mathrm{t}} = \mathbf{\Omega}\mathbf{C}\mathbf{\Omega}^{\mathrm{T}}, \tag{36a}$$

where $\mathbf{\Omega}$ is the transformation matrix defined by

$$\mathbf{\Omega} = \begin{bmatrix} \cos^2\theta & \sin^2\theta & -2\sin\theta\cos\theta \\ \sin^2\theta & \cos^2\theta & 2\sin\theta\cos\theta \\ \sin\theta\cos\theta & -\sin\theta\cos\theta & \cos^2\theta - \sin^2\theta \end{bmatrix}. \tag{36b}$$

Substituting (36) into (2), we can obtain the extensional, coupling and bending stiffness matrices as follows:

$$\begin{aligned} \boldsymbol{A} &= \begin{bmatrix} 135.66 & 94.26 & 97.32 \\ 94.26 & 135.66 & 97.32 \\ 97.32 & 97.32 & 106.8 \end{bmatrix} \text{(GPa mm)}, \qquad \boldsymbol{B} = \mathbf{0}\ (\text{GPa mm}^2), \\ \boldsymbol{D} &= \begin{bmatrix} 101.75 & 70.7 & 73.0 \\ 70.7 & 101.75 & 73.0 \\ 73.0 & 73.0 & 80.1 \end{bmatrix} (\text{GPa mm}^3). \end{aligned} \tag{37}$$

With the results obtained in (37) as our basic lamination properties, the forces and moments around the hole boundary can be calculated from the explicit solutions (33) through the detail flowchart shown in Fig. 3.

Because the laminate considered in this example is a symmetric laminate, the coupling stiffness matrix $\boldsymbol{B}$ is identical to zero, and hence the in-plane and plate bending problems are uncoupled. For the plate bending cases, the solutions have been obtained in (Hsieh and Hwu, 2002a). Table 1 shows that the moments around the circular hole boundary are identical to our previous results.

Example 2. A circular hole in an unsymmetric laminate

To see the coupling phenomenon, we now consider a [+45/0/+45/−45] unsymmetric laminate. By the way described in the previous example, we may obtain the extensional, coupling and bending stiffness matrices as follows:

$$\boldsymbol{A} = \begin{bmatrix} 274.46 & 96.98 & 32.44 \\ 96.98 & 144.71 & 32.44 \\ 32.44 & 32.44 & 113.7 \end{bmatrix} \text{(GPa mm)},$$

$$\boldsymbol{B} = \begin{bmatrix} 46.79 & -14.35 & 81.1 \\ -14.35 & -18.085 & 81.1 \\ 81.1 & 81.1 & -14.35 \end{bmatrix} (\text{GPa mm}^2),$$

$$\boldsymbol{D} = \begin{bmatrix} 272.37 & 158.01 & 10.81 \\ 158.01 & 229.12 & 10.81 \\ 10.81 & 10.81 & 180.3 \end{bmatrix} (\text{GPa mm}^3).$$

Fig. 4 shows the forces and moments distribution around the hole boundary under different loading conditions. From Fig. 4, we see that even the unsymmetric composite laminate is subjected to in-plane forces only or out-of-plane bending moment only, it will induce both of the bending moments and in-plane forces around the hole boundary, which is reasonable and expectable due to the existence of the coupling stiffnesses.

Example 3. Coupling effects of the laminates

The purpose of this example is to show the coupling effects of the anti-symmetric laminates through the gradual changing of the coupling stiffnesses B_{16} and B_{26}. Consider five different laminates whose extensional and bending stiffnesses are the same as

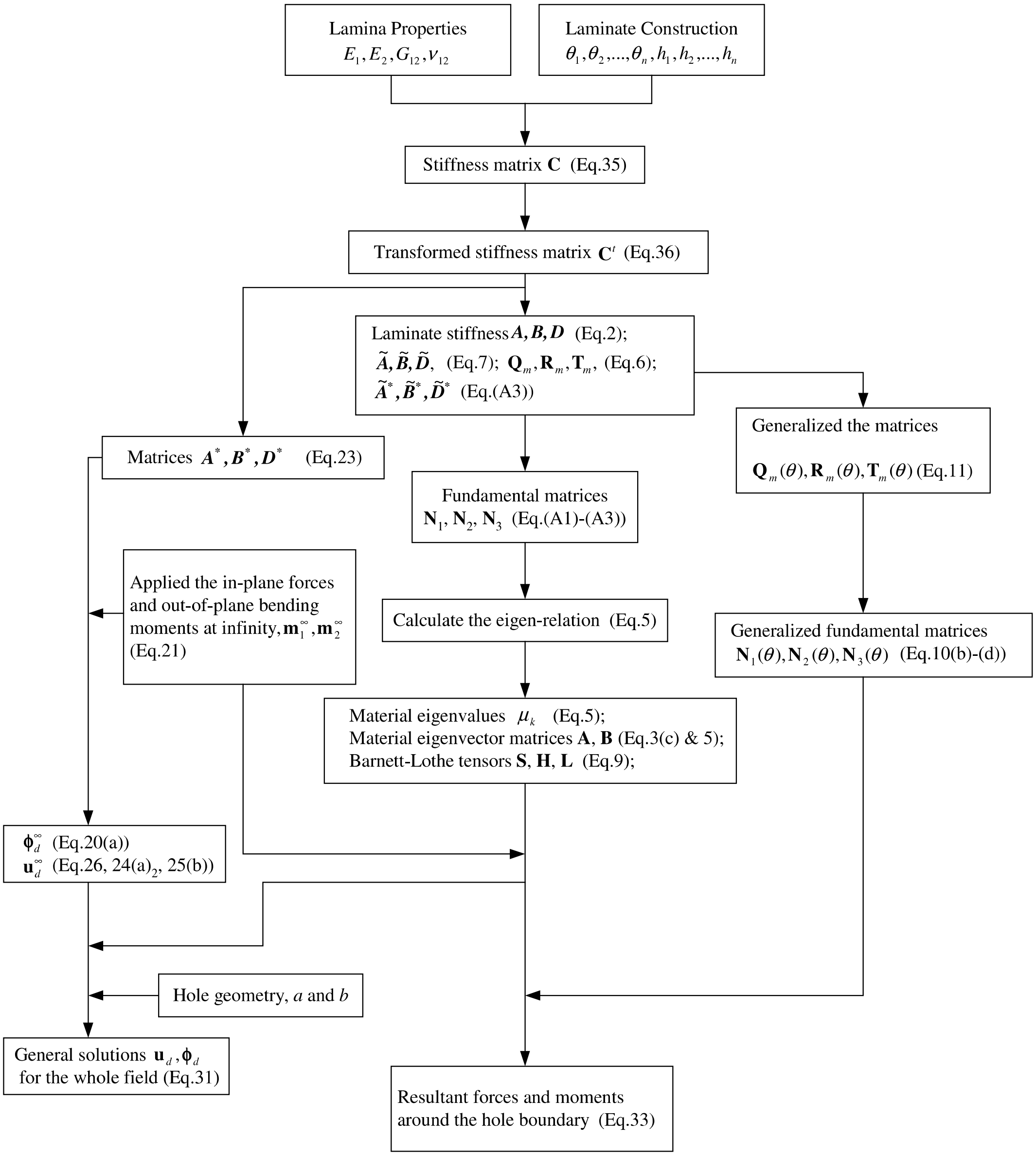


Fig. 3. Flowchart for the calculation of the deformations and stresses, and the resultant forces and moments around the hole boundary.

$$\boldsymbol{A} = \begin{bmatrix} 339.34 & 96.99 & 0 \\ 96.99 & 79.84 & 0 \\ 0 & 0 & 113.71 \end{bmatrix} \text{(GPa mm)},$$

$$\boldsymbol{D} = \begin{bmatrix} 452.46 & 129.32 & 0 \\ 129.32 & 106.42 & 0 \\ 0 & 0 & 151.61 \end{bmatrix} \text{(GPa mm}^3\text{)},$$

Table 1
Moments around the circular hole boundary of an unidirectional laminate subjected to out-of-plane bending moments $M_{11}^{\infty} = \hat{m}$

Moments Angle(φ)	$M_{ns}/\hat{m}$		$M_s/\hat{m}$	
	Present	Hsieh and Hwu (2002b)	Present	Hsieh and Hwu (2002b)
0°	−0.5477	−0.5476	−0.2938	−0.2937
30°	−0.5959	−0.5959	0.4760	0.4761
60°	−0.3160	−0.3160	1.0812	1.0813
90°	0.1602	0.1602	1.3943	1.3944
120°	0.5639	0.5638	1.7809	1.7814
150°	0.7920	0.7922	1.5973	1.5979
180°	−0.5477	−0.5476	−0.2938	−0.2937
210°	−0.5959	−0.5959	0.4760	0.4761
240°	−0.3160	−0.3160	1.0812	1.0813
270°	0.1602	0.1602	1.3943	1.3944
300°	0.5639	0.5638	1.7809	1.7814
330°	0.7920	0.7922	1.5973	1.5979

but their coupling stiffnesses are different each other as

Case 1: $\boldsymbol{B} = \boldsymbol{0}$ (GPa mm^2), (symmetric laminates)
Case 2:

$$\boldsymbol{B} = \begin{bmatrix} 0 & 0 & 40.52 \\ 0 & 0 & 15.665 \\ 40.52 & 15.665 & 0 \end{bmatrix}, \quad \text{(half as Case 3)}$$

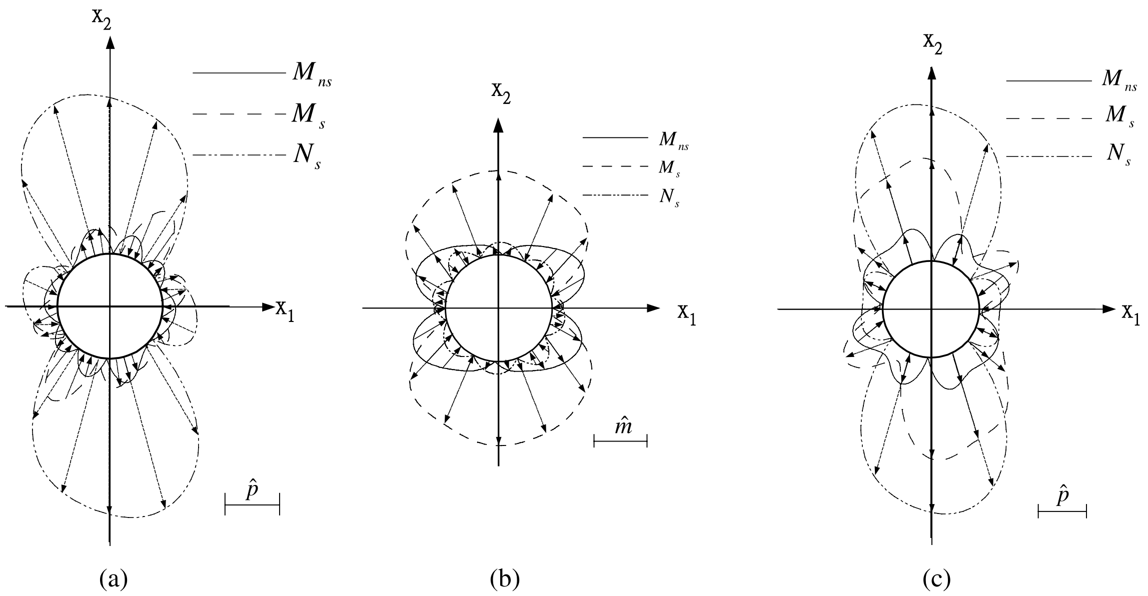


Fig. 4. Force and moment around the circular hole in an unsymmetric composite laminate under different loading conditions: (a) subjected to $N_{11}^{\infty} = \hat{p}$, (b) subjected to $M_{11}^{\infty} = \hat{m}$, and (c) subjected to $N_{11}^{\infty} = \hat{p}$, $M_{11}^{\infty} = \hat{m}$ ($\hat{m} = \hat{p} \times 1$).

Case 3:

$$\boldsymbol{B} = \begin{bmatrix} 0 & 0 & 81.04 \\ 0 & 0 & 31.33 \\ 81.04 & 31.33 & 0 \end{bmatrix}, \quad ([-30/+30/-30/+30] \text{ anti-symmetric laminates})$$

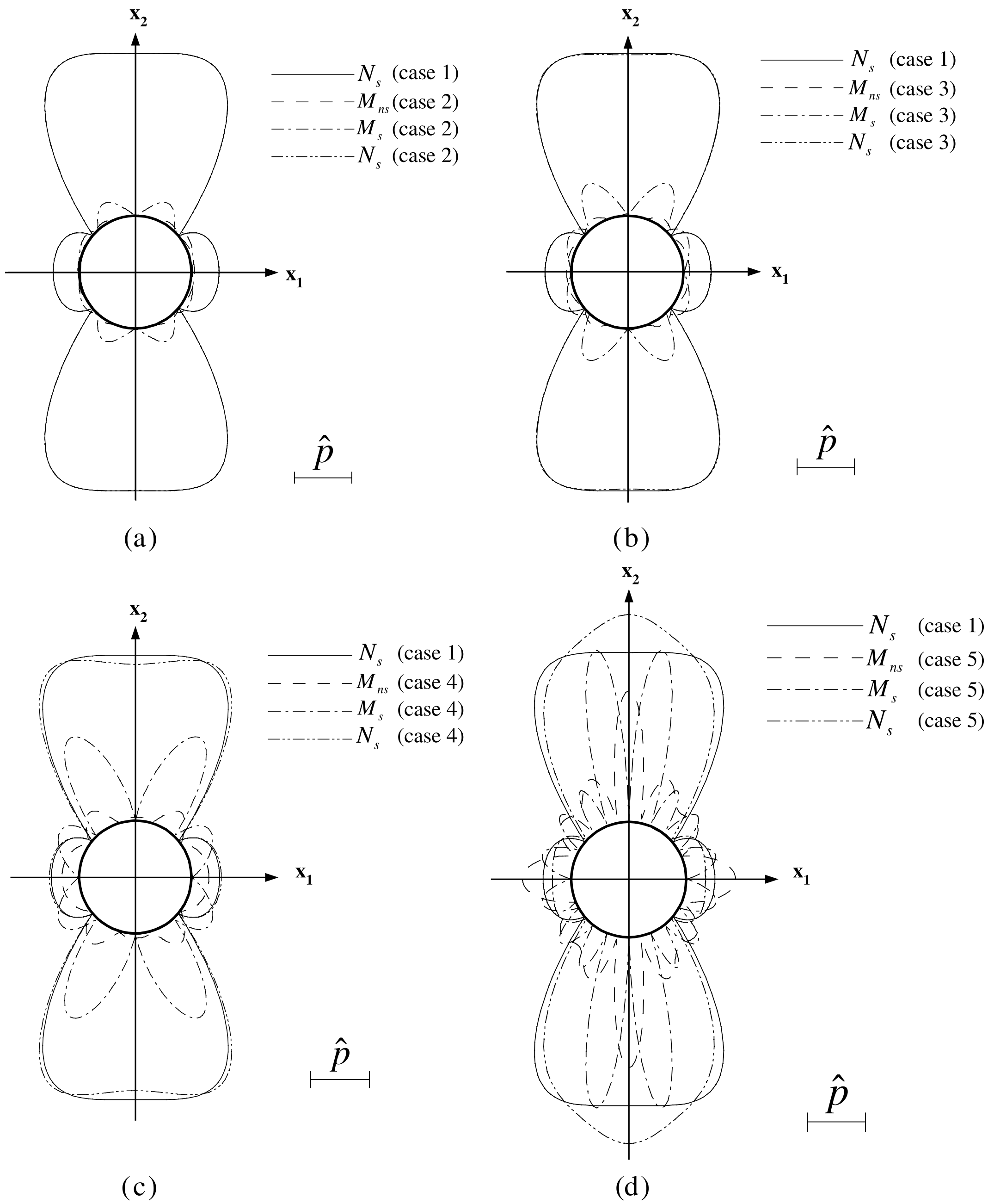


Fig. 5. Force and moment distribution around the hole boundaries of five different laminates subjected to in-plane forces $N_{11}^{\infty} = \hat{p}$ only.

Case 4:

$$\boldsymbol{B} = \begin{bmatrix} 0 & 0 & 162.08 \\ 0 & 0 & 62.66 \\ 162.08 & 62.66 & 0 \end{bmatrix}, \quad \text{(twice as Case 3)}$$

Case 5:

$$\boldsymbol{B} = \begin{bmatrix} 0 & 0 & 405.2 \\ 0 & 0 & 156.65 \\ 405.2 & 156.65 & 0 \end{bmatrix}. \quad \text{(five times as Case 3)}$$

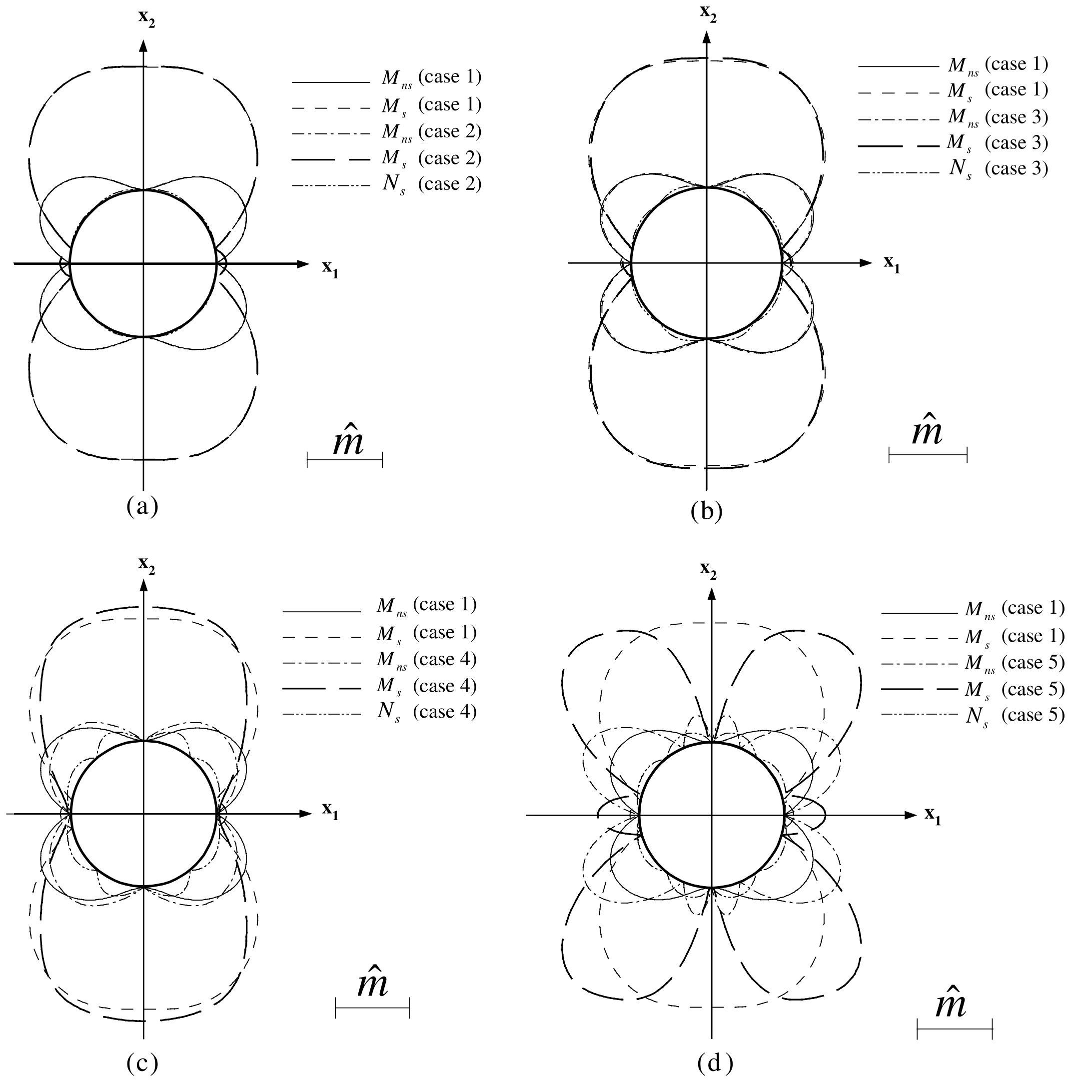


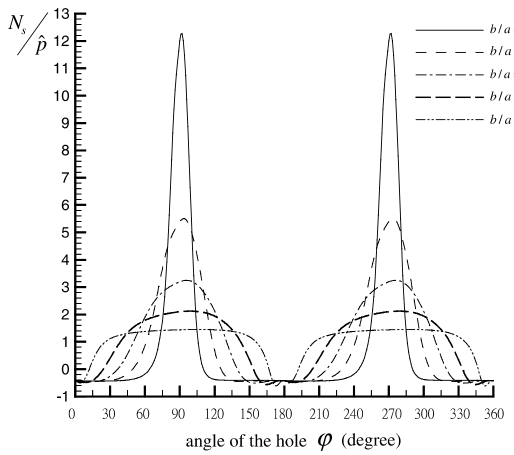
Fig. 6. Force and moment distribution around the hole boundaries of five different laminates subjected to out-of-plane bending moments $M_{11}^{\infty} = \hat{m}$ only.

In order to see the coupling effects more clearly, the in-plane force $N_{11}^{\infty} = \hat{p}$ and bending moment $M_{11}^{\infty} = \hat{m}$ are applied separately in Figs. 5 and 6. Only the in-plane force $\hat{p}$ is applied for the cases shown in Fig. 5. Thus, no bending moments can be induced for the symmetric laminates (Case 1), whereas both in-plane resultant forces and bending moment occur for all the other cases due to the addition of the coupling stiffnesses B_{16} and B_{26}. Fig. 5(a)–(c) show that the addition of B_{16} and B_{26} does not influence too much for the values of N_s, whereas the gradual increase of M_{ns} and M_s due to the gradual increase of B_{16} and B_{26} is quite obvious. Fig. 5(d) shows that not only the values of N_s, M_{ns} and M_s increase but also their distribution profile change a lot for Case 5 when the values of coupling stiffnesses B_{16} and B_{26} are increased to five times of Case 3.

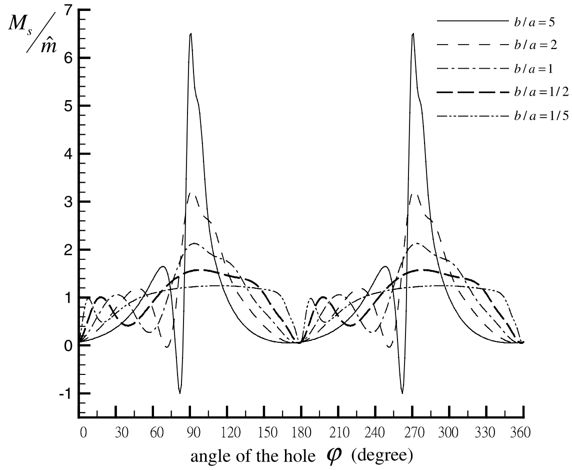
For the cases of the laminates subjected to out-of-plane bending moment $M_{11}^{\infty} = \hat{m}$ only, similar trend can be observed in Fig. 6(a)–(d).

Example 4. Shape effects of the holes

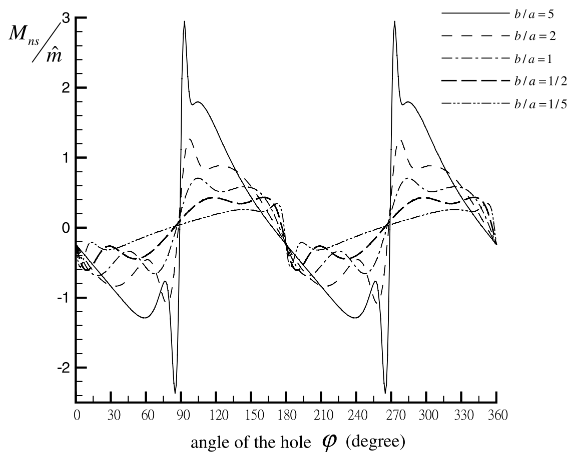
Because the solutions obtained in this paper are also valid for general elliptical hole, it is interesting to know the effects of hole shapes by just changing the ratio of minor axis to major axis. In this example, the



(a)



(b)



(c)

Fig. 7. Force and moment around the elliptical hole in an unsymmetric laminate subjected to $N_{11}^{\infty} = \hat{p}$, $M_{11}^{\infty} = \hat{m}$ ($\hat{m} = \hat{p} \times 1$).

laminate is considered to be the same as that of Example 2. Fig. 7(a)–(c) show the resultant force N_s, moment M_s and twist M_{ns} versus hole boundary angle φ for five different ratios, $b/a = 5, 2, 1, 1/2, 1/5$. From Fig. 7 we see that the maximum values of N_s, M_s and M_{ns} increase as the ratio b/a increases. Moreover, the maximum values of N_s, M_s and M_{ns} will locate at $\varphi = 90°$ and $270°$ when the ratio $b/a \geqslant 1$, which is conformable to our engineering intuition. Moreover, when the ratio b/a is getting higher, the elliptical hole will approach to a crack perpendicular to the applied force $\hat{p}$ and moment $\hat{m}$ ($\hat{m} = \hat{p} \times 1$). Therefore, the maximum values of N_s, M_s and M_{ns} will approach to infinity as the ratio approaches to infinity. On the other hand, for the ratio $b/a < 1$, the location of the maximum values has the tendency, although not quite clear, to shift to the places near $\varphi = 180°$ and $\varphi = 360°$, which is also reasonable from the viewpoint of horizontal crack to be its limiting case.

6. Conclusion

By applying a recent developed Stroh-like formalism for the coupled stretching–bending analysis of composite laminates, analytical solutions for elliptical holes in laminates subjected to in-plane forces (including in-plane shear forces) and/or out-of-plane bending moments (including twisting moments) are obtained explicitly in this paper. Because the Stroh-like formalism has been purposely organized into the form of Stroh formalism for two-dimensional analysis, the present field solutions have exactly the same form as those obtained for the two-dimensional problems. It is therefore expected that all the other unsolved coupled stretching–bending problems can be solved automatically if their corresponding two-dimensional problems have been solved, such as the inclusion problems (Hwu and Ting, 1989; Hwu and Yen, 1993).

Like the two-dimensional problems, although the field solutions can only be expressed in complex forms through the use of some identities the resultant forces and moments around the hole boundary have been obtained in real forms. In addition to the explicit solutions presented in this paper, several numerical examples are done to verify our results, and to study the coupling effects of the laminates and the shape effects of the holes. The results are well agree to the existing solutions for the symmetric laminates and the engineering intuition for the unsymmetric laminates.

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Appendix A

The explicit expressions of the fundamental matrix **N** for the displacement-based formalism can be expressed as

$$\mathbf{N}_1 = \frac{1}{\tilde{\tilde{\Delta}}}\begin{bmatrix} X_{11} & X_{12} & Y_{13} & Y_{14} \\ X_{21} & 0 & Y_{23} & Y_{24} \\ 0 & 0 & 0 & X_{43} \\ \widetilde{B}^*_{12} & 0 & X_{34} & X_{44} \end{bmatrix}, \qquad \mathbf{N}_2 = \frac{1}{\tilde{\tilde{\Delta}}}\begin{bmatrix} Y_{11} & Y_{12} & 0 & X_{14} \\ Y_{12} & Y_{22} & 0 & X_{24} \\ 0 & 0 & 0 & 0 \\ X_{14} & X_{24} & 0 & \widetilde{A}^*_{11} \end{bmatrix},$$

$$\mathbf{N}_3 = \frac{1}{\tilde{\tilde{\Delta}}}\begin{bmatrix} -\widetilde{D}^*_{22} & 0 & X_{31} & X_{41} \\ 0 & 0 & 0 & 0 \\ X_{31} & 0 & Y_{33} & Y_{34} \\ X_{41} & 0 & Y_{34} & Y_{44} \end{bmatrix}, \tag{A.1}$$

where

$$
\begin{aligned}
&\widetilde{\Delta} = \widetilde{B}^*_{12}\widetilde{B}^*_{12} + \widetilde{A}^*_{11}\widetilde{D}^*_{22},\\
&X_{11} = \widetilde{A}^*_{16}\widetilde{D}^*_{22} + \widetilde{B}^*_{12}\widetilde{B}^*_{62}, \qquad X_{12} = -(\widetilde{A}^*_{11}\widetilde{D}^*_{22} + \widetilde{B}^*_{12}\widetilde{B}^*_{12}), \qquad X_{14} = \widetilde{A}^*_{11}\widetilde{B}^*_{62} - \widetilde{A}^*_{16}\widetilde{B}^*_{12},\\
&X_{21} = \widetilde{A}^*_{12}\widetilde{D}^*_{22} + \widetilde{B}^*_{12}\widetilde{B}^*_{22}, \qquad X_{24} = \widetilde{A}^*_{11}\widetilde{B}^*_{22} - \widetilde{A}^*_{12}\widetilde{B}^*_{12},\\
&X_{31} = \widetilde{B}^*_{11}\widetilde{D}^*_{22} - \widetilde{B}^*_{12}\widetilde{D}^*_{12}, \qquad X_{34} = -(\widetilde{A}^*_{11}\widetilde{D}^*_{12} + \widetilde{B}^*_{11}\widetilde{B}^*_{12}),\\
&X_{41} = 2(\widetilde{B}^*_{16}\widetilde{D}^*_{22} - \widetilde{B}^*_{12}\widetilde{D}^*_{26}), \qquad X_{43} = \widetilde{A}^*_{11}\widetilde{D}^*_{22} + \widetilde{B}^*_{12}\widetilde{B}^*_{12}, \qquad X_{44} = -2(\widetilde{A}^*_{11}\widetilde{D}^*_{26} + \widetilde{B}^*_{16}\widetilde{B}^*_{12}),\\
&Y_{11} = \widetilde{A}^*_{11}\widetilde{B}^{*2}_{62} + \widetilde{A}^*_{66}\widetilde{B}^{*2}_{12} + \widetilde{A}^*_{11}\widetilde{A}^*_{66}\widetilde{D}^*_{22} - \widetilde{A}^{*2}_{16}\widetilde{D}^*_{22} - 2\widetilde{A}^*_{16}\widetilde{B}^*_{12}\widetilde{B}^*_{62},\\
&Y_{12} = \widetilde{A}^*_{11}\widetilde{B}^*_{22}\widetilde{B}^*_{62} + \widetilde{A}^*_{26}\widetilde{B}^{*2}_{12} + \widetilde{A}^*_{11}\widetilde{A}^*_{26}\widetilde{D}^*_{22} - \widetilde{A}^*_{12}\widetilde{A}^*_{16}\widetilde{D}^*_{22} - \widetilde{A}^*_{12}\widetilde{B}^*_{12}\widetilde{B}^*_{62} - \widetilde{A}^*_{16}\widetilde{B}^*_{12}\widetilde{B}^*_{22},\\
&Y_{13} = \widetilde{A}^*_{16}\widetilde{B}^*_{12}\widetilde{D}^*_{12} + \widetilde{B}^{*2}_{12}\widetilde{B}^*_{61} + \widetilde{A}^*_{11}\widetilde{B}^*_{61}\widetilde{D}^*_{22} - \widetilde{A}^*_{16}\widetilde{B}^*_{11}\widetilde{D}^*_{22} - \widetilde{A}^*_{11}\widetilde{B}^*_{62}\widetilde{D}^*_{12} - \widetilde{B}^*_{11}\widetilde{B}^*_{12}\widetilde{B}^*_{62},\\
&Y_{14} = 2(\widetilde{A}^*_{16}\widetilde{B}^*_{12}\widetilde{D}^*_{26} + \widetilde{B}^{*2}_{12}\widetilde{B}^*_{66} + \widetilde{A}^*_{11}\widetilde{B}^*_{66}\widetilde{D}^*_{22} - \widetilde{A}^*_{16}\widetilde{B}^*_{16}\widetilde{D}^*_{22} - \widetilde{A}^*_{11}\widetilde{B}^*_{62}\widetilde{D}^*_{26} - \widetilde{B}^*_{12}\widetilde{B}^*_{16}\widetilde{B}^*_{62}),\\
&Y_{22} = \widetilde{A}^*_{11}\widetilde{B}^{*2}_{22} + \widetilde{A}^*_{22}\widetilde{B}^{*2}_{12} + \widetilde{A}^*_{11}\widetilde{A}^*_{22}\widetilde{D}^*_{22} - \widetilde{A}^{*2}_{12}\widetilde{D}^*_{22} - 2\widetilde{A}^*_{12}\widetilde{B}^*_{12}\widetilde{B}^*_{22},\\
&Y_{23} = \widetilde{A}^*_{12}\widetilde{B}^*_{12}\widetilde{D}^*_{12} + \widetilde{B}^{*2}_{12}\widetilde{B}^*_{21} + \widetilde{A}^*_{11}\widetilde{B}^*_{21}\widetilde{D}^*_{22} - \widetilde{A}^*_{12}\widetilde{B}^*_{11}\widetilde{D}^*_{22} - \widetilde{A}^*_{11}\widetilde{B}^*_{22}\widetilde{D}^*_{12} - \widetilde{B}^*_{11}\widetilde{B}^*_{12}\widetilde{B}^*_{22},\\
&Y_{24} = 2(\widetilde{A}^*_{12}\widetilde{B}^*_{12}\widetilde{D}^*_{26} + \widetilde{B}^{*2}_{12}\widetilde{B}^*_{26} + \widetilde{A}^*_{11}\widetilde{B}^*_{26}\widetilde{D}^*_{22} - \widetilde{A}^*_{12}\widetilde{B}^*_{16}\widetilde{D}^*_{22} - \widetilde{A}^*_{11}\widetilde{B}^*_{22}\widetilde{D}^*_{26} - \widetilde{B}^*_{12}\widetilde{B}^*_{16}\widetilde{B}^*_{22}),\\
&Y_{33} = \widetilde{A}^*_{11}\widetilde{D}^{*2}_{12} + 2\widetilde{B}^*_{11}\widetilde{B}^*_{12}\widetilde{D}^*_{12} - \widetilde{B}^{*2}_{11}\widetilde{D}^*_{22} - \widetilde{B}^{*2}_{12}\widetilde{D}^*_{11} - \widetilde{A}^*_{11}\widetilde{D}^*_{11}\widetilde{D}^*_{22},\\
&Y_{34} = 2(\widetilde{B}^*_{12}\widetilde{B}^*_{16}\widetilde{D}^*_{12} + \widetilde{A}^*_{11}\widetilde{D}^*_{12}\widetilde{D}^*_{26} + \widetilde{B}^*_{11}\widetilde{B}^*_{12}\widetilde{D}^*_{26} - \widetilde{A}^*_{11}\widetilde{D}^*_{16}\widetilde{D}^*_{22} - \widetilde{B}^{*2}_{12}\widetilde{D}^*_{16} - \widetilde{B}^*_{11}\widetilde{B}^*_{16}\widetilde{D}^*_{22}),\\
&Y_{44} = 4(\widetilde{A}^*_{11}\widetilde{D}^{*2}_{26} + 2\widetilde{B}^*_{12}\widetilde{B}^*_{16}\widetilde{D}^*_{26} - \widetilde{B}^{*2}_{16}\widetilde{D}^*_{22} - \widetilde{B}^{*2}_{12}\widetilde{D}^*_{66} - \widetilde{A}^*_{11}\widetilde{D}^*_{22}\widetilde{D}^*_{66}),
\end{aligned}
\tag{A.2}
$$

in which

$$\widetilde{\boldsymbol{A}}^* = \boldsymbol{A}^{-1}, \qquad \widetilde{\boldsymbol{B}}^* = -\boldsymbol{A}^{-1}\boldsymbol{B}, \qquad \widetilde{\boldsymbol{D}}^* = \boldsymbol{D} - \boldsymbol{B}\boldsymbol{A}^{-1}\boldsymbol{B}. \tag{A.3}$$

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